Solutions 3 MAT 5190 M. ALVO

5.27. a) $$f_{X_{(i)}|X_{(j)}}(u,v) = \frac{f_{X_{(i)},X_{(j)}}(u,v)}{f_{X_{(j)}}(v)}$$

For $i<j$,

$$= \frac{\frac{n!}{(i-1)!(j-1-i)!(n-j)!} f_X(u) f_X(v) \left[F_X(u)\right]^{i-1} \left[F_X(v)-F_X(u)\right]^{j-1-i} \left[1-F_X(v)\right]^{n-j}}{\frac{n!}{(j-1)!(n-j)!} f_X(v) \left[F_X(v)\right]^{j-1} \left[1-F_X(v)\right]^{n-j}}$$

(3)

$$= \frac{(j-1)!}{(i-1)!(j-1-i)!} \frac{f_X(u)}{f_X(v)} \cdot \left[\frac{F_X(u)}{F_X(v)}\right]^{i-1} \left[1-\frac{F_X(u)}{F_X(v)}\right]^{j-i-1} \quad u<v$$

For $j<i$, $u>v$ similarly we get

$$f_{X_{(i)}|X_{(j)}}(u,v) = \frac{(n-j)!}{(i-j-1)!(n-i)!} \frac{f_X(u)}{1-F_X(v)} \cdot \left[\frac{F_X(u)-F_X(v)}{1-F_X(v)}\right]^{i-j-1} \left[1-\frac{F_X(u)-F_X(v)}{1-F_X(v)}\right]^{n-i}$$

b) We have that $$f_{R,V}(r,v) = \frac{n(n-1)}{a^n} r^{n-2} \qquad 0<r<a,\ r/2<v<a-r/2$$

(2)

$$f_R(r) = \frac{n(n-1)\, r^{n-2}}{a^n}(a-r) \qquad 0<r<a$$

$$\therefore\ f_{R|V}(r|v) = \frac{1}{a-r} \qquad r/2<v<a-r/2$$

Hilroy

5.36 a) We use Theorem 4.4.3 which states that

$$E[X] = E\left[E[X|Y]\right]$$

and Theorem 4.4.7 (p 167)

$$Var[X] = E\left[Var(X|Y)\right] + Var\left[E[X|Y]\right]$$

In this example we have

$$E[Y|N=n] = 2n \qquad (\text{mean of } \chi^2_{2n})$$

$$Var[Y|N=n] = 2(2n) = 4n \qquad (\text{Var of } \chi^2_{2n})$$

$$\therefore \quad E[Y] = E[2N] = 2\theta \qquad (\text{mean of Poisson}(\theta))$$

$$Var\left[E[Y|N]\right] = Var[2N] = 4\theta \qquad (\text{Variance of Poisson } \theta)$$

$$E\left[Var[Y|N]\right] = E[4N] = 4\theta \qquad (\text{mean of Poisson}(\theta))$$

$$\therefore \quad Var[X] = 8\theta$$

b) We will calculate the moment generating function of

$$Z = \frac{Y - EY}{\sqrt{Var Y}} = \frac{Y - 2\theta}{\sqrt{8\theta}}$$

The mgf of $\mathcal{P}(\theta)$ is $e^{\theta(e^t - 1)}$

The mgf of χ^2_{2n} is $(1-2t)^{-n}$

$$M_Z(t) = E\left[e^{tZ}\right] = E\left[e^{(t/\sqrt{8\theta})Y}\right] \cdot e^{-\left(\frac{2\theta}{\sqrt{8\theta}}\right)t}. \quad \text{Set } a = \frac{1}{\sqrt{8\theta}}$$

Now $$E\left[e^{atY}\right] = E\,E\left[e^{atY} \,|\, N=n\right] = E\left[(1-2at)^{-N}\right]$$

$$= \sum_{n=0}^{\infty} (1-2at)^{-n} e^{-\theta} \frac{\theta^n}{n!} = e^{-\theta} \exp\left[\frac{\theta}{1-2at}\right]$$

Consider $$\log M_Z(t) = -\theta + \frac{\theta}{1-2at} - \frac{2\theta}{\sqrt{8\theta}} t \longrightarrow \frac{t^2}{2} \text{ as } \theta \to \infty. \text{ Q.E.D.}$$

5.4.2 a) $f_X(x) = \beta(1-x)^{\beta-1} \qquad 0 < x < 1 \quad \beta > 0$

$F_X(x) = 1-(1-x)^{\beta}$

$\therefore \quad P\left(X_{(n)} \le 1-x\right) = \left(1-x^{\beta}\right)^n$

$\therefore \quad P\left(1-X_{(n)} \le x\right) = \left(1-x^{\beta}\right)^n$

Choose $x = \dfrac{\varepsilon}{n^{\nu}}$. We know $\left(1+\dfrac{e}{n}\right)^n \longrightarrow e^{e}$ as $n \to \infty$.

$\therefore \quad 1-\dfrac{\varepsilon^{\beta}}{n^{\beta\nu}} = 1-\dfrac{\varepsilon^{\beta}}{n}\left(\dfrac{n}{n^{\beta\nu}}\right)$

Choose $\nu \ni \dfrac{n}{n^{\beta\nu}} = 1 \Rightarrow \dfrac{1}{n^{\beta\nu-1}} = 1 \Rightarrow \beta\nu = 1$

$\therefore \quad n^{1/\beta}\left(1-X_{(n)}\right)$ will have cdf $e^{-x^{\beta}}$

b) $X \sim \exp(1) \Rightarrow f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{o.w.} \end{cases}$

$F(x) = 1-e^{-x}$

$P\left(X_{(n)} \le x\right) = \left(1-e^{-x}\right)^n$

Set $x = u + a_n$

$\therefore \quad 1-e^{-x} = 1-e^{-u}\cdot e^{a_n} = 1-\dfrac{e^{-u}}{n}\cdot n\,e^{a_n}$

Choose a_n such that $n\,e^{a_n} = 1 \Rightarrow a_n = -\log n$

5.49 a) Trivial

b) Set $X = g(U) = -\log \frac{1-U}{U}$

Then $g^{-1}(x) = \frac{1}{1+e^{-x}}$

and $f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}$ which is the logistic density $0 < x < \infty$

c) If X is logistic (μ, β) then

$f_X(x) = \frac{1}{\beta} f_Y\left(\frac{-y+\mu}{\beta}\right)$ where $Y \sim$ logistic $(0,1)$

So to generate X, first generate $U \sim \text{Unif}(0,1)$

and set $Y = \beta \log \frac{U}{1-U} + \mu$.

Hilroy